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On Connected Transversals to Subgroups Whose Order is a Product of Two Primes

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We consider finite groups which have connected transversals to subgroups whose order is a product of two primes p and q. We investigate those values of p and q for which the group is soluble. We can show that the solubility of the group follows if $q = 2$ and $p \le 61$, $q = 3$ and $p \le 31$, $q = 5$ and $p \le 11$. We then apply our results on loop theory and we show that if the inner mapping group of a finite loop has order pq where p and q are as above then the loop is soluble.

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1. INTRODUCTION

Let G be a finite group and H a subgroup of G. If A and B are two left transversals to H in G and $[A, B] \le H$ then we say that A and B are H-connected in G. This concept was introduced in [5] and it was used to characterize multiplication groups of loops. We then started to investigate the relation between the solubility of G and the structure of H. In [6] we showed that if H is abelian then G is soluble and we also managed to show in [4] that G is soluble if $|H| = 6$ (thus covering the smallest non-abelian case). In this paper we consider the following general problem: If $|H| = pq$ where p and q are prime numbers, does it then follow that G is soluble? We are able to prove this in the following cases: $q = 2$ and $p \le 61$, $q = 3$ and $p \le 31$, $q = 5$ and $p = 11$. In our proofs we use the properties of finite simple groups as given in the *Atlas of Group Theory* [1].

We have already mentioned that the theory of connected transversals is linked to loop theory. We also apply here our results on loop theory and we show that if the order of the inner mapping group of a loop is pq (where p and q are as above) then the loop is soluble.

Our notation in group theory is standard. Perhaps we should point out that given a group G and its subsets A and B we denote by $[A, B]$ the subgroup which is generated by all possible commutators $[a, b] = a^{-1}b^{-1}ab$ ($a \in A$, $b \in B$). If H is a subgroup of a group G then H_G denotes the core of H in G (the largest normal subgroup of G contained in H). Basic facts about connected transversals, loops and their multiplication groups can be found in [3, 5, 7]. In this paper we consider finite loops and groups only.

2. SOME LEMMAS

Here we introduce some results which are needed later in Section 3. In the first four lemmas we assume that A and B are H-connected transversals in G.

LEMMA 2.1. *If $C \subseteq A \cup B$ and $T = \langle H, C\rangle$ then $C \subseteq T_G$.*

LEMMA 2.2. *If $H_G = 1$ and $[A, B] = 1$ then A and B are isomorphic subgroups of G.*

LEMMA 2.3. *If H is abelian then G is soluble.*

For the proofs, see [5, Lemma 2.5] and [6, Lemma 2.3 and Theorem 3.4].

LEMMA 2.4. *Let $a \in A$ and assume that the set $E = \{[a, b] \mid b \in B\} \subseteq H$ has t elements. If $|H| = k$, then $[G : C_G(a)] \le kt$.*

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PROOF. Let h be a fixed element from E and write $T(a,h) = \{b \in B \mid a^{-1}b^{-1}ab = h\}$. If $b, c \in T(a,h)$, then $bc^{-1} \in C_G(a)$ and therefore $b \in C_G(a)c$. Thus $T(a,h) \subseteq C_G(a)b(h)$ where $b(h)$ is a fixed element from $T(a,h)$ (depending on h). Now $B = \cup T(a,h)$ where h goes through all the elements of E. Thus $B \subseteq C_G(a)\{b(h) \mid h \in E\}$ and since $G = BH$ the result follows. $\square$

We still need a result on factorized groups and for the proof we need the well-known Burnside's theorem.

THEOREM 2.1 (BURNSIDE'S THEOREM). *A finite group $G = AK$ is not a non-abelian simple group if $Z(A) \neq 1$ and $[G : A] = p^n$ for some prime p.*

LEMMA 2.5. *Let $G = AH$ where A is abelian and H is a subgroup of G such that $|H| = pq$ where p and q are prime numbers and $p > q$. Then G is soluble.*

PROOF. We let G be a minimal counterexample and conclude easily that G is simple. Assume first that A is maximal in G. Now A is not normal in G, hence we have a conjugate $A^g \neq A$. Thus $G = \langle A, A^g \rangle$. Clearly, $N_G(A) = A$ and $A \cap A^g = 1$. We conclude that G is a Frobenius group with Frobenius complement A and thus G is soluble (see [2, p. 499]).

So assume that $A < M < G$. We must have $[G : M] = p$ and thus $M = AQ$ where Q is a q-subgroup of H. If q divides $|A|$ then we have a q-element $x \in A$ such that $x \in C_M(Q)$. Thus $Z(M) > 1$ and from Burnside's theorem it follows that G is not simple. Thus q does not divide $|A|$. Now we have a conjugate A^f which is not contained in M, hence $M \cap A^f = 1$. Then $|MA^f| = |A|^2 q \leq |A|pq$, hence $|A| \leq p$. If $|A| = p$ then G is soluble. Further, $[G : H] = |A| < p$ is not possible. The proof is complete. $\square$

3. MAIN THEOREM

We can now prove our main theorem. In the proof we reduce our problem to finite simple groups whose orders have an upper bound based on combinatorial calculations. After this we use the properties of those finite simple groups by using the *Atlas of Finite Groups* [1].

THEOREM 3.1. *Let G be a finite group, $H \leq G$ and $|H| = pq$ where p and q are prime numbers and let A and B be two H-connected transversals in G. Then G is soluble at least in the following cases:*

(a) $p = q$,
(b) *q is not a factor of $p - 1$,*
(c) $q = 2$ *and* $p \leq 61$,
(d) $q = 3$ *and* $p \leq 31$,
(e) $q = 5$ *and* $p = 11$.

PROOF. We first point out that if $p = q$ or if q is not a factor of $p - 1$ then H is abelian (in fact, cyclic in the latter case) and from Lemma 2.3 it follows that G is soluble. Thus we may assume that $p > q$, q divides $p - 1$ and H is non-abelian.

Let G be a minimal counterexample. If H is not a maximal subgroup in G then there exists a subgroup T such that $G > T > H$. By Lemma 2.1, $T_G > 1$ and clearly G/T_G and T are soluble, hence G is soluble. Thus we may assume that H is maximal in G. It is also clear that $H_G = 1$.

If N is a nontrivial proper normal subgroup of G then $G = NH$. Write $H = PQ$ where P is the normal Sylow p-subgroup and Q is of order q. Now $N_G(P) = H$. Since $N \cap H$ is normal in H, it follows that $N \cap H \leq P$. Denote $E = NP$. Then $N_E(P) = P$ and E is a Frobenius group with Frobenius complement P. By using the properties of Frobenius groups

([2, p. 499]) we conclude that E is soluble. It follows that N and G are soluble. Thus G has no nontrivial normal subgroups and we may assume that G is a finite simple group.

Since there exist H-connected transversals A and B in G (that is, $[A, B] \leq H$), we look at the commutators $[a, b]$ (these commutators are either trivial, p-elements or q-elements).

[(1) Assume that $[A, B] = 1$. If $A \neq B$ then A and B are isomorphic subgroups of G by Lemma 2.2. If $E = \langle A, B\rangle$ is a proper subgroup of G then A and B are normal in E, hence $E = AB$. Clearly, $A \cap B \leq Z(E)$. Since $G = EH$ it follows that $A/A \cap B$ is cyclic, hence A is abelian (also B is abelian). Thus by Lemma 2.5, G is soluble. If $E = G$ then A is normal in G, hence $A = 1$ and $G = H$. Finally, if $A = B$ then A is an abelian group. Again it follows from Lemma 2.5, that G is soluble.
Thus we may assume that $[A, B]$ is not trivial.

(2) Now assume that $[a, b]$ is a p-element for some $a \in A$, $b \in B$. If $\langle a, b\rangle$ is a proper subgroup of G, then $\langle a, b\rangle$ is a Frobenius group with P as a Frobenius complement. Thus $\langle a, b\rangle = D = KP$ where K is the Frobenius kernel. Here $D' \leq K$, hence $[a, b] \in K \cap P = 1$ and this is a contradiction. Thus we may assume that $\langle a, b\rangle = G$. Now $[G : C_G(a)] \leq (pq)^2$ and $[G : C_G(b)] \leq (pq)^2$ by Lemma 2.4. Since $C_G(a) \cap C_G(b) = Z(G) = 1$ we conclude that $|G| \leq (pq)^4$. Now we have produced our first upper bound for the order of G (there is more to come).

(3) Now we shall consider the case that $[A, B]$ is not trivial and if $a \in A$ and $b \in B$ then $[a, b]$ is either trivial or a q-element of H. By Lemma 2.4 we get $[G : C_G(d)] \leq pq(pq - p + 1)$ for each $d \in A \cup B$.
If $[a, B]$ contains q-elements from at least two different q-subgroups of H (say, $[a, b]$ and $[a, c]$) then $G = \langle a, b, c\rangle$ and $|G| \leq (pq)^3(pq - p + 1)^3$.
Next assume that $[a, B] \leq Q(a)$ where $Q(a)$ is a q-subgroup of H (depending on a) and $[A, b] \leq Q(b)$ where $Q(b)$ is a q-subgroup of H (depending on b). Now $[G : C_G(d)] \leq pq^2$ for each $d \in A \cup B$. If $[A, B]$ is not contained in one single q-subgroup of H then G is generated by four elements from $A \cup B$ and $|G| \leq p^4q^8$.
Thus we can next assume that $1 < [A, B] \leq Q$ where Q is a q-subgroup of H. If $aH = bH$ and $b^{-1}a \notin Q$ then again G is generated by four elements from $A \cup B$ and $|G| \leq p^4q^8$.
Thus we may assume that $b^{-1}a \in Q$ whenever $aH = bH$ and we are in a situation which is similar to the one described in the proof of Theorem 3.1 of [5]. We proceed as in that proof and we conclude that $A = B$.

(4) Thus we assume that $1 < [A, A] \leq Q$. If $a, b \in A$ then we have a unique $g(a, b) \in A$ such that $abH = g(a, b)H$. We now write $h(a, b) = g(a, b)^{-1}ab \in H$. If $h(a, b)$ is not in Q and if $[a, d] \neq 1$ for $d \in A$ then G is generated by four elements from A and $|G| \leq p^4q^8$. Thus assume that $[a, A] = 1$ (likewise, we may assume that $[b, A] = [g(a, b), A] = 1$). Thus $[G : C_G(a)] \leq pq$ (the same is true for the centralizers of b and $g(a, b)$). Since $[e, f] \neq 1$ for some $e, f \in A$ we have $G = \langle e, f, a, b, g(a, b)\rangle$, hence $|G| \leq (pq^2)^2(pq)^3 = p^5q^7$. If the elements $h(a, b) \in Q$ for all $a, b \in A$ then we can proceed as in the proof of Theorem 3.1 in [5] and we conclude that A is an abelian group. Thus $G = AH$ and it follows from Lemma 2.5 that G is soluble.

Thus we now face the following situation: G is a finite simple group, H is maximal in G, $|H| = pq$, q divides $p - 1$ and by combining all the upper bounds we get

$$|G| \leq \max\{p^5q^7, (pq)^3(pq - p + 1)^3\}.$$

Now we use the *Atlas of Finite Groups* [1] where the list of maximal subgroups of finite simple groups is complete up to the order 495 766 656 000 (for the sporadic Conway group Co_3). By using the upper bounds for the order of G we see that the following cases have to be checked:

(6, PSL(2, 5)) (we first write the order of H and then those finite simple groups which have a maximal subgroup of this order)
(10, PSL(2, 5))
(14, PSL(2, 8), PSL(2, 13) and Sz(8))
(22, PSL(2, 23))
(26, PSL(2, 25) and PSL(2, 27))
(34, PSL(2, 16))
(62, PSL(2, 32) and Sz(32))
(21, PSL(2, 7))
(39, PSL(3, 3) and $U_3(4)$)
(57, PSL(3, 7) and $U_3(8)$)
(93, PSL(3, 5))
(55, PSL(2, 11)).

Since Vesanen [8, 9] has shown that it is not possible for those special linear groups PSL$(2, q)$ which appear in our list to have connected transversals to the corresponding subgroups, we may concentrate on the following seven cases.

(i) Consider the Suzuki group Sz(8) which has a maximal subgroup of order 14 and whose order is 29 120. We apply Lemma 2.4 and it follows that $|G_G(d)| > 148$ for any $d \in A \cup B$. Since this is larger than any of the centralizers in Sz(8), we have a contradiction.

(ii) The Suzuki group Sz(32) has a maximal subgroup of order 62 and the order of the group is 32 537 600. Here again we can use Lemma 2.4 and we see that $|C_G(d)| > 8464$ for every $d \in A \cup B$. These centralizers are also too large.

(iii) The group PSL(3, 3) has a maximal subgroup of order 39 and the order of the group is 5616. We know that $a^{-1}b^{-1}ab \in H$ and thus $a^b \in aH$. If $a^b = a^c$ then $bc^{-1} \in C_G(a)$. Now $|B| = 144$ and thus we place $144-39$ elements in the set $S = C_G(a) - \{1\}$. In PSL(3, 3) we have $|C_G(x)| \leq 13$ for each nontrivial $x \in G$. Thus $|S| \leq 12$ and therefore there exists $y \in S$ such that $y = b_1{c_1}^{-1} = b_2{c_2}^{-1}$. Now $b_1 = yc_1$ and $b_2 = yc_2$. If $a \in A$ then $[a, yc_i] = [a, c_i][a, y]^{c_i} \in H$, hence $[a, y] \in H^{{c_1}^{-1}} \cap H^{{c_2}^{-1}} = F$. Since $N_G(H) = H$, we conclude that $|F| \leq 3$. Now we know that $[a, y] \in F$ for every $a \in A$ and thus we put 144 commutators in F. If $[a_1, y] = [a_2, y]$ then $a_1a_2^{-1} \in C_G(y)$. It follows that there exists $z \in F$ such that $z = [a_1, y] = \cdots = [a_k, y]$ where $k \geq 48$. Thus $C_G(y)$ has at least 47 elements which is not possible.

In the remaining cases we can apply the same technique as in (iii).

(iv) Now $|U_3(4)| = 62\,400$ and $|H| = 39$. Thus $|C_G(y)| \geq 534$.

(v) The order of PSL(3, 7) is 1 876 896, $|H| = 57$ and the order of the centralizer is greater than 10 976.

(vi) The order of $U_3(8) = 5\,515\,776$, $|H| = 57$ and the order of the centralizer is greater than 32 256.

(vii) The order of PSL(3, 5) is 372 000, $|H| = 93$ and the order of the centralizer is greater than 1333.

In all cases we have centralizers which are too large. This is our final contradiction and the proof of the theorem is complete. □

4. Loop Theory

The relation between multiplication groups of loops and connected transversals is given by

THEOREM 4.1. *A group G is isomorphic to the multiplication group of a loop iff there exists a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

For the proof, see [5, Theorem 4.1]. Quite recently the following theorem has been proved by Vesanen [10, Theorem 1].

THEOREM 4.2. *If Q is a finite loop whose multiplication group is soluble then Q is soluble.*

By combining Theorem 3.1 with these results we immediately have

THEOREM 4.3. *If Q is a finite loop whose inner mapping group has order pq where p and q are two prime numbers as in Theorem 3.1 then Q is a soluble loop.*

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